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A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE--ETC(U)
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TECHNICAL MEMORANDUM 592

6 A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE DISCONTINUITIES, APPENDIX A.

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F O R E W O R D

This memorandum covers only a small phase of the problem of instrumentation for signal enhancement research, and has been prepared primarily for internal distribution to aid others at NEL who may be interested in related problems. ~~Only a limited distribution outside of the Laboratory is contemplated.~~ Work to February 1963 is covered.

INTRODUCTION

Part I of this series, consisting of three sections:

1. Mathematical Background.

2. Theoretical Development.

3. Treatment of Single Frequency Input.

was issued as TM-279.

Part II, consisting of four sections:

4. Application to Single Sideband Signals.

5. Effect of Doppler.

6. Lower Sideband and Double Sideband Operation.

7. Effect of Finite Duration of Sample.

was issued as TM-302.

the Mathematical background section

Appendix A systematizes the material from ~~Section 1~~, somewhat, and extends the formal treatment to form a broader mathematical background for future developments in this series.

APPENDIX A

Extension of Mathematical Background

Some additional Fourier transform relations are needed as background for further development of the treatment of cyclic phenomena with periodic phase discontinuities. For this purpose it is convenient to return to fundamentals, with the result that some of the relations in sections 1 and 2 may be developed or restated. In what follows it will be assumed that the functions discussed are well-behaved functions in the sense that their Fourier transforms and inverse Fourier transforms exist. This includes the generalized functions as defined and discussed by Lighthill[6].

The definition of $X(f)$ as the Fourier transform of $x(t)$, and the equivalent inverse Fourier transform relation, may be expressed respectively in the forms *:

$$X(f) = \mathfrak{F}_{tf}\, x(t) = \int x(t) \exp(-2\pi i f t)dt \quad , \qquad \text{A-1a}$$

$$x(t) = \mathfrak{F}_{ft}^{-1}\, X(f) = \int X(f) \exp(2\pi i f t)df \quad . \qquad \text{A-1b}$$

Interchange of f and t in equations (A-1) results in

$$X(t) = \mathfrak{F}_{ft}\, x(f) = \int x(f) \exp(-2\pi i f t)df \quad , \qquad \text{A-2a}$$

$$x(f) = \mathfrak{F}_{tf}^{-1}\, X(t) = \int X(t) \exp(2\pi i f t)dt \quad . \qquad \text{A-2b}$$

Replacement of f by $-f$ in (A-2) gives

$$X(t) = \int x(-f) \exp(2\pi i f t)df = \mathfrak{F}_{ft}^{-1}\, x(-f) \quad , \qquad \text{A-3a}$$

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* In this paper all integrations are assumed to be taken over the significant region of the integrand unless otherwise indicated, and are thus effectively equivalent to integration from $-\infty$ to $+\infty$ even though no limits are shown.

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6 M. J. Lighthill, "An Introduction to Fourier Analysis and Generalized Functions," Cambridge Univ. Press, 1958.

$$x(-f) = \int X(t) \exp(-2\pi ift)dt = \mathfrak{F}_{tf}\ X(t) \quad . \qquad \text{A-3b}$$

Similarly, replacement of f by $-f$ and t by $-t$ in (A-1) leads to

$$X(-f) = \int x(-t) \exp(-2\pi ift)dt = \mathfrak{F}_{tf}\ x(-t) \quad , \qquad \text{A-4a}$$

$$x(-t) = \int X(-f) \exp(2\pi ift)df = \mathfrak{F}_{ft}^{-1}\ X(-f) \quad . \qquad \text{A-4b}$$

The spectrum is represented by the Fourier transform of the corresponding waveform, and conversely the waveform is given by the inverse Fourier transform of the spectrum.

Thus from A-1 we have

$$\text{spec } x(t) = \mathfrak{F}_{tf}\ x(t) = X(f) \quad , \qquad \text{1-1}$$

$$\text{wave } X(f) = \mathfrak{F}_{ft}^{-1}\ X(f) = x(t) \qquad \text{1-2}$$

as noted in section 1.

Also (A-3b) and (A-4a) respectively give

$$\text{spec } X(t) = \mathfrak{F}_{tf}\ X(t) = x(-f) \qquad \text{1-16}$$

and

$$\text{spec } x(-t) = \mathfrak{F}_{tf}\ x(-t) = X(-f) \quad . \qquad \text{1-17}$$

The above waveform - spectrum relations, along with some other special relations from section 1 are listed in Table 1 for convenient reference. Most of these relations are listed by Woodward on p. 28 of Reference 5, or may be obtained quite simply from others which he lists. Other waveform-spectrum pairs in Table 1 may be obtained from Lighthill (p. 45 of ref. 6) with some changes in notation. Thus the signum function,

$$\operatorname{sgn} t = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \qquad \text{A-5a}$$

may be used to define the step function

$$\operatorname{step} t = [1 + \operatorname{sgn} t]/2 \quad . \qquad \text{A-5b}$$

The corresponding spectra are given as

$$\operatorname{spec} \operatorname{sgn} t = 1/\pi i f \qquad \text{A-6}$$

and

$$\operatorname{spec} \operatorname{step} t = [\delta(f) + 1/\pi i f]/2 \quad . \qquad \text{A-7}$$

Let us represent the Fourier transformation of the functional combination $\phi[t, u(t),\ldots,w(t)]$ by $\Psi[f, U(f), \ldots, W(f)]$. We may thus write

$$\operatorname{spec} \phi[t, u(t)] = \Psi[f, U(f)] \qquad \text{A-8a}$$

and corresponding to (1-16)

$$\operatorname{spec} \Psi[t, U(t)] = \phi[-f, u(-f)], \qquad \text{A-8b}$$

or, equivalently,

$$\operatorname{spec} \Psi[-t, U(-t)] = \phi[f, u(f)] \quad . \qquad \text{A-8c}$$

It is seen that replacement of $U(t)$ in (A-8b) by $u(t)$ gives

$$\operatorname{spec} \Psi[t, u(t)] = \phi[-f, U(f)] \quad . \qquad \text{A-9a}$$

Similarly, replacement of $U(-t)$ in (A-8c) by $u(t)$ gives

$$\operatorname{spec} \Psi[-t, u(t)] = \phi[f, U(f)] \quad . \qquad \text{A-9b}$$

It follows from (A-9a) and (A-9b) that if either

$$\phi[-f,\ U(f)] = \phi[f,\ U(f)] \qquad \text{A-10a}$$

or

$$\Psi[-t,\ u(t)] = \Psi[t,\ u(t)] \qquad \text{A-10b}$$

is true, then

$$\text{spec}\ \Psi[t,\ u(t)] = \phi[f,\ U(f)] \quad . \qquad \text{A-11a}$$

But if (A-11a) is true it follows from (A-9a) and (A-9b) that both (A-10a) and (A-10b) must be true. Hence, if either (A-10a) or (A-10b) is true, the other is true also. It follows that (A-10a) and (A-10b) are equivalent relations and either relation may be taken as a necessary and sufficient condition for the validity of (A-11a). By replacing f by t and U by u we may put (A-10a) in the equivalent form

$$\phi[-t,\ u(t)] = \phi[\ t,\ u(t)] \qquad \text{A-11b}$$

which may then also be taken as a necessary and sufficient condition for the validity of (A-11a).

The method used in setting up the relations (A-8) to (A-11) could have been used equally well for functional combinations involving any finite number of functions. Thus it is seen that if the spectrum of a waveform $\phi[t,\ u(t),\ldots,w(t)]$ has been obtained in the form

$$\text{spec}\ \phi[t,\ u(t),\ldots,w(t)] = \Psi[f,\ U(f),\ldots,W(f)] \quad , \qquad \text{A-12a}$$

then the complementary relation may be written

$$\text{spec}\ \Psi[t,\ u(t),\ldots,w(t)] = \phi[f,\ U(f),\ldots,W(f)] \quad , \qquad \text{A-12b}$$

provided only that

$$\phi[-t, u(t),\ldots,w(t)] = \phi[t, u(t),\ldots,w(t)] \quad . \qquad \text{A-12c}$$

When (A-12a) takes on the simplified form

$$\text{spec } \phi[u(t),\ldots,w(t)] = \Psi[U(f),\ldots,W(f)], \qquad \text{A-13a}$$

t may be considered as entering explicitly into ϕ as a zero power. Relation (A-12c) is thus satisfied and it follows that

$$\text{spec } \Psi[u(t),\ldots,w(t)] = \phi[U(f),\ldots,W(f)] \quad . \qquad \text{A-13b}$$

Actually (A-13) can be made somewhat more general in form. The above arguments can be used to show that when (A-12a) takes on the form

$$\text{spec } \phi[u(t),\ldots,w(t)] = \Psi[f, U(f),\ldots,W(f)], \qquad \text{A-14a}$$

then (A-12c) is satisfied and it follows that

$$\text{spec } \Psi[t, u(t),\ldots,w(t)] = \phi[U(f),\ldots,W(f)] \quad . \qquad \text{A-14b}$$

Similarly, it may be seen that if (A-12a) takes on the form

$$\text{spec } \phi[t, u(t),\ldots,w(t)] = \Psi[U(f),\ldots,W(f)] \quad , \qquad \text{A-15a}$$

then

$$\text{spec } \Psi[u(t),\ldots,w(t)] = \phi[f, U(f),\ldots,W(f)] \qquad \text{A-15b}$$

is also true.

The usefulness of (A-13) is exemplified in (1-23) and (1-24) since either of these latter two relations follows immediately from the other by application of (A-13). Similarly (1-28) follows from (1-27) by application of (A-14), and (1-27)

would have followed from (1-28) by application of (A-15) if (1-28) had been obtained first. Equation (1-44) may be taken as a special example of (A-12a), since for this example t enters explicitly in ψ only in the $\delta(t)$, which is even, thereby satisfying (A-12c). That is

$$\text{rep}_S\,[u(t)\ \text{rep}_R\ \delta(-t)] = \text{rep}_S\,[u(t)\ \text{rep}_R\ \delta(t)] \quad . \qquad \text{A-16}$$

It follows that (A-12b) must hold for this special example, and (A-12) may thus be applied to (1-44) to obtain (1-45). Other examples appear in what follows

Equation (1-27) may be set into (1-23) to obtain

$$\text{spec}\,[u(t)\ \text{rep}_S\ v(t)] = |\,1/S\,|\ \text{conv}\,[U(f),\ V(f)\ \text{rep}_{1/S}\ \delta(f)]$$

$$= |\,1/S\,| \int U(f-f_1)\ V(f_1)\ \ \text{rep}_{1/S}\ \ \delta(f_1)\ df_1$$

$$= |\,1/S\,| \int U(f-f_1)\ V(f_1)\ \Sigma_r\ \delta(f_1-r/S)\ df_1$$

$$= |\,1/S\,|\ \Sigma_r \int U(f-f_1)\ V(f_1)\ \delta(f_1-r/S)\ df_1$$

$$= |\,1/S\,|\ \Sigma_r\ U(f-r/S)\ V(r/S) \qquad \text{A-17a}$$

Application of (A-13) then gives

$$\text{spec}\ \ \Sigma_r\ u(t-rS)\ \ v(rS) = |\,1/S\,|\ U(f)\ \text{rep}_{1/S}\ V(f) \qquad \text{A-17b}$$

Similarly (1-27) may be set into (A-17a) to obtain

$$\text{spec}\,[\text{rep}_R\ u(t)\ \text{rep}_S\ v(t)] = |\,1/RS\,|\ \Sigma_r\ U(f-r/S)\ V(r/S)\ \text{rep}_{1/R}\ \delta(f-r/S)$$

$$= |\,1/RS\,|\Sigma_r\ U(f-r/S)\ V(r/S)\ \Sigma_n\ \delta(f-n/R-r/S)$$

$$= |\,1/RS\,|\Sigma_r\ V(r/S)\ \ \Sigma_n\ \ U(f-r/S)\ \ \delta(f-n/R-r/S)$$

$$= |\,1/RS\,|\Sigma_r\ V(r/S)\ \Sigma_n\ \ U(n/R)\ \ \delta(f-n/R-r/S)$$

$$= \left| 1/RS \right| \Sigma_n \Sigma_r \; U(n/R) \; V(r/S) \; \delta(f-n/R-r/S) \; . \qquad \text{A-18a}$$

Application of (A-12) to (A-18a) then gives

$$\text{spec } \Sigma_n \Sigma_r \; u(nR) \; v(rS) \; \delta(t-nR-rS) = \left| 1/RS \right| \text{rep}_{1/R} U(f) \; \text{rep}_{1/S} V(f) \; . \qquad \text{A-18b}$$

The above and some other pairs of relations illustrating the utility of (A-12) and (A-13) are listed in Table 2. Here (A-19a) results when (A-18a) is set back into (1-23), (A-20a) when (A-17a) is set into (1-27), and (A-21a) when (A-20a) is set into (1-23). Similarly (A-22a) and (A-23a) respectively are obtained when (A-17a) is set into (1-28) and when (1-28) is set into (A-17a). In each of these cases the corresponding relation denoted by the use of a "b" instead of an "a" in the equation number, is obtained from the "a" relation by application of (A-12) or one of its simplified versions (A-13), (A-14) and (A-15). The final relation (A-24) is obtained when (1-14) is set into (1-27).

The above processes could be continued for the treatment of more and more complicated waveforms. The list in Table 2 is adequate to illustrate the method, however. Other relations will be developed as the need arises. It may be worth noting that (A-18) suggests an extension of the definition of the comb notation (1-10) to two functions. Thus

$$\text{comb}_{R,S} \left[u(t), v(t) \right] \equiv \Sigma_n \Sigma_r \; u(nR) \; v(rS) \; \delta(t-nR-rS) \qquad \text{A-25}$$

makes it possible to express (A-18) in the form

$$\text{spec} \left[\text{rep}_R \; u(t) \; \text{rep}_S \; v(t) \right] = \left| 1/RS \right| \text{comb}_{1/R,1/S} \left[U(f), V(f) \right], \qquad \text{A-26a}$$

$$\text{spec comb}_{R,S} \left[u(t), v(t) \right] = \left| 1/RS \right| \left[\text{rep}_{1/R} \; U(f) \; \text{rep}_{1/S} \; V(f) \right] \; , \qquad \text{A-26b}$$

in analogy with (1-25) and (1-26). More generally, one may extend the definition of the comb function to any finite number of functions by writing.

$$\mathrm{comb}_{R,S,\ldots,T}\,[u(t), v(t),\ldots,w(t)]$$

$$= \Sigma_n \Sigma_r \ldots \Sigma_s\, u(nR)\, v(rS) \ldots w(sT)\quad \delta(t-nR-rS-\ldots-sT) \quad . \qquad \text{A-27}$$

It may then be shown that

$$\mathrm{spec}\,[\mathrm{rep}_R\, u(t)\ \mathrm{rep}_S\, v(t) \ldots \mathrm{rep}_T\, w(t)] \qquad \text{A-28a}$$

$$= \;|\, 1/RS\ldots T \,|\; \mathrm{comb}_{1/R,\, 1/S,\ldots,1/T}\; [U(f), V(f),\ldots,W(f)] \quad ,$$

$$\mathrm{spec}\ \mathrm{comb}_{R,S,\ldots,T}\ [u(t), v(t),\ldots,w(t)]$$

$$= \;|\, 1/RS\ldots T \,|\; \mathrm{rep}_{1/R}\, U(f)\ \mathrm{rep}_{1/S}\, V(f)\ldots \mathrm{rep}_{1/T}\, W(f) \quad . \qquad \text{A-28b}$$

The definition of convolution in (1-3) may be extended to any number of functions. Thus for three functions we may write

$$\mathrm{conv}\,[u(t), v(t), w(t)] = \int\int u(t_1)\, v(t_2)\, w(t-t_1-t_2)\, dt_1\, dt_2$$

$$= \int\int u(t_1)\, v(t-t_1-t_3)\;\; w(t_3)\, dt_1\, dt_3$$

$$= \int\int u(t-t_2-t_3)\;\; v(t_2)\, w(t_3)\, dt_2\;\; dt_3 \qquad \text{A-29}$$

Now, equation (1-23) may be applied to itself to obtain

$$\mathrm{spec}\,[u(t)\, v(t)\, w(t)] = \mathrm{conv}\;\left\{U(f), \mathrm{conv}\;[V(f), W(f)]\right\}$$

$$= \int U(f_1) \int V(f_2)\, W(f-f_1-f_2)\;\; df_2\, df_1$$

$$= \int\int U(f_1)\, V(f_2)\, W(f-f_1-f_2)\;\; df_1\, df_2$$

$$= \mathrm{conv}\,[U(f), V(f), W(f)] \qquad \text{A-30a}$$

Application of (A-13) then gives

$$\text{spec conv } [u(t), v(t), w(t)] = U(f)\,V(f)\,W(f) \quad . \qquad \text{A-30b}$$

The extension of these last results to products and convolutions of any finite number of functions is fairly obvious. Caution is indicated, however, when more than one of the functions multiplied together contain δ functions. This caution applies to any of the relations in which the products of general functions is indicated.

TABLE 1

Equation No.	Waveform	Spectrum
1-1	$u(t)$	$U(f)$
1-16	$U(t)$	$u(-f)$
1-17	$u(-t)$	$U(-f)$
1-18	$u^*(t)$	$U^*(-f)$
1-19	$u'(t)$	$2\pi i f\ U(f)$
1-22	$Au(t) + Bv(t)$	$AU(f) + BV(f)$
1-20	$u(t-\tau)$	$U(f)\ \exp(-2\pi i f\tau)$
1-21	$u(t/T)$	$\|T\|\ U(fT)$
1-30	$u[(t-\tau)/T]$	$\|T\|\ U(fT)\ \exp(-2\pi i f\tau)$
1-13	$\mathrm{rect}(t)$	$\mathrm{sinc}(f)$
1-38	$\mathrm{rect}(t/T)$	$\|T\|\ \mathrm{sinc}(fT)$
1-31	$\exp(2\pi i\varphi t)$	$\delta(f-\varphi)$
1-14	$\delta(t)$	1
1-15	$\delta(t-\tau)$	$\exp(-2\pi i f\tau)$
1-39	$u(t)\ \exp(2\pi i\varphi t)$	$U(f-\varphi)$
1-40	$\mathrm{rect}(t)\ \exp(2\pi i\varphi t)$	$\mathrm{sinc}(f-\varphi)$
1-41	$\mathrm{rect}(t/T)\ \exp(2\pi i\varphi t + i\beta)$	$\|T\|\ \mathrm{sinc}[(f-\varphi)T]\ \exp(i\beta)$
1-35	$\cos(2\pi\varphi t + \beta)$	$(1/2)[\delta(f-\varphi)\ \exp(i\beta) + \delta(f+\varphi)\ \exp(-i\beta)]$
1-37	$\cos^2(\pi\varphi t)$	$(1/4)[\delta(f-\varphi) + 2\delta(f) + \delta(f+\varphi)]$
1-32	1	$\delta(f)$
A-6	$\mathrm{sgn}(t)$	$1/\pi i f$
A-7	$\mathrm{step}(t)$	$[\delta(f) + 1/\pi i f]/2$

TABLE 2

Equation No.	Waveform	Spectrum
1-23	$u(t)\ v(t)$	conv $[U(f),\ V(f)]$
1-24	conv $[u(t),\ v(t)]$	$U(f)\ V(f)$
1-27	$\text{rep}_R\ u(t)$	$\lvert 1/R \rvert\ U(f)\ \text{rep}_{1/R}\ \delta(f)$
1-28	$u(t)\ \text{rep}_R\ \delta(t)$	$\lvert 1/R \rvert\ \text{rep}_{1/R}\ U(f)$
1-44	$\text{rep}_S[u(t)\ \text{rep}_R\ \delta(t)]$	$\lvert 1/RS \rvert\ \text{rep}_{1/R} U(f)\ \text{rep}_{1/S} \delta(f)$
1-45	$\text{rep}_R\ u(t)\ \text{rep}_S \delta(t)$	$\lvert 1/RS \rvert\ \text{rep}_{1/S}\ [U(f)\ \text{rep}_{1/R}\ \delta(f)]$
A-13a	$\phi[u(t),\ v(t),\ w(t)]$	$\psi[U(f),\ V(f),\ W(f)]$
A-13b	$\psi[u(t),\ v(t),\ w(t)]$	$\phi[U(f),\ V(f),\ W(f)]$
A-17a	$u(t)\ \text{rep}_S\ v(t)$	$\lvert 1/S \rvert\ \Sigma_r\ U(f-r/S)\ V(r/S)$
A-17b	$\Sigma_r u(t-rS)\ v(rS)$	$\lvert 1/S \rvert\ U(f)\ \text{rep}_{1/S}\ V(f)$
A-18a	$\text{rep}_R\ u(t)\ \text{rep}_S\ v(t)$	$\lvert 1/RS \rvert\ \Sigma_n \Sigma_r\ U(n/R)\ V(r/S)\ \delta(f-n/R-r/S)$
A-18b	$\Sigma_n \Sigma_r\ u(nR) v(rS) \delta(t-nR-rS)$	$\lvert 1/RS \rvert\ \text{rep}_{1/R}\ U(f)\ \text{rep}_{1/S}\ V(f)$
A-19a	$u(t)\ \text{rep}_S\ v(t)\ \text{rep}_T\ w(t)$	$\lvert 1/ST \rvert\ \Sigma_r \Sigma_s\ U(f-r/S-s/T)\ V(r/S)\ W(s/T)$
A-19b	$\Sigma_r \Sigma_s u(t-rS-sT) v(rS) w(sT)$	$\lvert 1/ST \rvert\ \text{rep}_{1/S}\ V(f)\ \text{rep}_{1/T}\ W(f)$
A-20a	$\text{rep}_R[u(t)\ \text{rep}_S\ v(t)]$	$\lvert 1/RS \rvert\ \Sigma_r\ U(f-r/S)\ V(r/S)\ \text{rep}_{1/R}\ \delta(f)$
A-20b	$\Sigma_r\ u(t-rS)\ v(rS)\ \text{rep}_R\ \delta(t)$	$\lvert 1/RS \rvert\ \text{rep}_{1/R}\ [U(f)\ \text{rep}_{1/S} V(f)]$
A-21a	$u(t)\ \text{rep}_S\ [v(t)\ \text{rep}_T\ w(t)]$	$\lvert 1/ST \rvert\ \Sigma_r \Sigma_s U(f-r/S)\ V(r/S-s/T)\ W(s/T)$
A-21b	$\Sigma_r \Sigma_s u(t-rS) v(rS-sT) w(sT)$	$\lvert 1/ST \rvert\ U(f)\ \text{rep}_{1/S}[V(f)\ \text{rep}_{1/T}\ W(f)]$
A-22a	$u(t)\ \text{rep}_S\ v(t)\ \text{rep}_T\ \delta(t)$	$\lvert 1/ST \rvert\ \Sigma_r\ V(r/S)\ \text{rep}_{1/T} U(f-r/S)$
A-22b	$\Sigma_r\ v(rS)\ \text{rep}_T\ u(t-rS)$	$\lvert 1/ST \rvert\ U(f)\ \text{rep}_{1/S} V(f)\ \text{rep}_{1/T}\ \delta(f)$
A-23a	$u(t)\ \text{rep}_T[v(t)\ \text{rep}_S\ \delta(t)]$	$\lvert 1/ST \rvert\ \Sigma_s\ U(f-s/T)\ \Sigma_r V(s/T-r/S)$
A-23b	$\Sigma_s\ u(t-sT)\ \Sigma_r\ v(sT-rS)$	$\lvert 1/ST \rvert\ U(f)\ \text{rep}_{1/T}[V(f)\ \text{rep}_{1/S}\ \delta(f)]$
A-24	$\text{rep}_R\ \delta(t)$	$\lvert 1/R \rvert\ \text{rep}_{1/R}\ \delta(f)$